AN UPPER LIMIT FOR THE VALUE OF A DETERMINANT.

By Thomas Muir, LL.D., F.R.S.

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1. In 1885 I was led to establish the theorem, *If* s_r^2 *be the sum of the squares of the elements of the* r*th row of a determinant* δ, *and* S_r^2 *be the sum of the squares of the elements of the corresponding row of the adjugate determinant* Δ, *then* $\delta \leqq s_r S_r$: and by means of it succeeded in proving for Sir William Thomson (afterward Lord Kelvin) the further inequality

$$\delta \leqq s_1 s_2 s_3 \dots$$

Taking, for example, the case where δ is of the fourth order, we obtain by using the former theorem four times

$$s_1 S_1 . s_2 S_2 . s_3 S_3 . s_4 S_4 \geqq \delta^4 ,$$

and therefore by Cauchy's theorem regarding the adjugate determinant

$$s_1 s_2 s_3 s_4 . S_1 S_2 S_3 S_4 \geqq \delta . \Delta .$$

From this the second theorem results at once.

Although it was agreed at the time that the latter theorem should be published in the *Educational Times,* it did not actually appear until 1901.*

2. It was in 1893 that the subject first assumed importance, M. Hadamard having in that year drawn the attention of mathematicians to it by means of two different papers.† His fundamental result, which is an extension of the theorem just mentioned so as to include determinants with complex elements, may be formally enunciated thus : *If* s_r^2 *be the sum*

* See *Educ. Times*, liv., p. 83, or *Math. from Educ. Times* (2), i., pp. 52, 53. The date of the theorem was there given from memory as being 1886. It should have been 1885. Lord Kelvin's letter approving of publication and remarking on the proof has since been recovered, and is dated "Nov. 12/85." [This letter has been duly shown to me as President of the Society.—S. S. Hough.]

† Hadamard, J., Résolution d'une question relative aux déterminants. *Bull. des Sci. math.* (2), xvii., pp. 240–246.

Hadamard, J., Sur le module maximum que puisse atteindre un déterminant. *Comptes-rendus . . . Acad. des Sci.* (Paris), cxi., pp. 1500–1501.

of the squares of the moduli of the elements of the r*th row of a determinant* δ, *then*

$$|\delta| \leqq s_1 s_2 s_3 \ldots$$

The original proof is neither short nor simple, the method followed being that known as "mathematical induction." From this some important conclusions are drawn, not the least interesting being those which are shown to link themselves on to a special class of determinants studied long before by Sylvester, and styled by him "inversely orthogonal determinants." *

3. In 1902 the subject acquired a still greater importance because of the intimate connection which it was found to have with Fredholm's equation.† This brought Professor Wirtinger to bestow attention on it, with the result that in 1907 he published ‡ a fresh proof of the fundamental theorem, his mode of procedure being to apply the ordinary Lagrangian rule for finding by differentiation the extreme values of a function whose variables are connected by equations of condition. This proof, though claimed to be shorter, and though having, of course, its own points of interest, cannot be said to be essentially simpler than that of Hadamard.

4. In these circumstances it seems desirable to point out, as I now propose to do, that my original method of treating the special case of the theorem is equally applicable when the elements of the determinant are complex quantities. Further, this having been done, it will readily appear that a fresh and simple presentment of the whole subject follows therefrom in a very natural way.

5. Denoting the determinant

$$\begin{vmatrix} a_1 + a'_1 i & a_2 + a'_2 i & a_3 + a'_3 i \\ b_1 + b'_1 i & b_2 + b'_2 i & b_3 + b'_3 i \\ c_1 + c'_1 i & c_2 + c'_2 i & c_3 + c'_3 i \end{vmatrix} \text{ by } \mu,$$

and its adjugate by $|A_1 + A'_1 i \quad B_2 + B'_2 i \quad C_3 + C'_3 i|$ or M, we have for $r = 1, 2, 3$,

$$\begin{aligned} \mu &= (a_r + a'_r i)(A_r + A'_r i) + (b_r + b'_r i)(B_r + B'_r i) + (c_r + c'_r i)(C_r + C'_r i) \\ &= (a_r A_r - a'_r A'_r + b_r B_r - b'_r B'_r + c_r C_r - c'_r C'_r) + (a_r A'_r + a'_r A_r + b_r B'_r + b'_r B_r \\ &\qquad + c_r C'_r + c'_r C_r)i \\ &= (\Sigma a_r A_r - \Sigma a'_r A'_r) + (\Sigma a_r A'_r + \Sigma a'_r A_r)i. \qquad \text{(I.)} \end{aligned}$$

* Sylvester, J. J., Thoughts on inverse orthogonal matrices, . . . *Philos. Magazine* (4), xxxiv. pp. 461–475 ; or *Collected Math. Papers*, ii., pp. 615–628.

† Fredholm, I., Sur une classe de transformations rationnelles. *Comptes-rendus . . . Acad. des Sci.* (Paris), cxxxiv., pp. 219–222, 1561–1564.

Fredholm, I., Sur une classe d'équations fonctionnelles. *Acta Math.*, xxvii., pp. 365–390.

‡ Wirtinger, W., Zum Hadamardschen Determinantensatz. *Monatshefte f. Math. u. Phys.*, xviii., pp. 158–160 ; or *Bull. des Sci. math.* (2), xxxi., pp. 175–179.

But whatever a_r, A_r, a'_r, A'_r, ... may be

$$\begin{Vmatrix} a_r+a'_ri & b_r+b'_ri & c_r+c'_ri \\ A_r-A'_ri & B_r-B'_ri & C_r-C'_ri \end{Vmatrix} \cdot \begin{Vmatrix} a_r-a'_ri & b_r-b'_ri & c_r-c'_ri \\ A_r+A'_ri & B_r+B'_ri & C_r+C'_ri \end{Vmatrix} \geqq 0,$$

because the left-hand member is the sum of three terms, each of which is the product of a complex quantity by its conjugate. Hence

$$\begin{vmatrix} a_r^2+a_r'^2+b_r^2+b_r'^2+c_r^2+c_r'^2 & (\Sigma a_rA_r-\Sigma a'_rA'_r)+(\Sigma a_rA'_r+\Sigma a'_rA_r)i \\ (\Sigma a_rA_r-\Sigma a'_rA'_r)-(\Sigma a_rA'_r+\Sigma a'_rA_r)i & A_r^2+A_r'^2+B_r^2+B_r'^2+C_r^2+C_r'^2 \end{vmatrix} \geqq 0,$$

or

$$(\Sigma a_r^2+\Sigma a_r'^2)(\Sigma A_r^2+\Sigma A_r'^2) \geqq (\Sigma a_rA_r-\Sigma a'_rA'_r)^2+(\Sigma a_rA'_r+\Sigma a'_rA_r)^2. \qquad \text{(II.)}$$

From (I.) and (II.) it follows that for $r=1, 2, 3$

$$(\Sigma a_r^2+\Sigma a_r'^2)(\Sigma A_r^2+\Sigma A_r'^2) \geqq |\mu|^2;$$

so that if s_r^2, S_r^2 stand for $\Sigma a_r^2+\Sigma a_r'^2$, $\Sigma A_r^2+\Sigma A_r'^2$, we have

$$s_1S_1.s_2S_2.s_3S_3 \geqq |\mu|^3,$$

and $\therefore$

$$s_1s_2s_3.S_1S_2S_3 \geqq |\mu|.|M|.$$

Now any reasons for $s_1s_2s_3$ being $<|\mu|$ would be equally effective in showing that $S_1S_2S_3<|M|$, and would thus by multiplication entail a result at variance with that just reached. Hence finally

$$s_1s_2s_3 \geqq |\mu|,$$

as was to be proved.

6. If μ' be what μ becomes on writing $-i$ for i, then $s_1^2s_2^2s_3^2$ is evidently the diagonal term of the determinant got by multiplying μ column-wise by μ': and the result of the preceding paragraph is that $\mu\mu'$ *is not greater than its own principal diagonal term, the product-determinant being obtained by column-wise multiplication, i.e.*

$$\ngtr (a_1^2+a_1'^2+b_1^2+b_1'^2+c_1^2+c_1'^2)(a_2^2+a_2'^2+b_2^2+b_2'^2+c_2^2+c_2'^2)(a_3^2+\ldots).$$

Of course we could prove in similar manner that $\mu\mu'$ *is not greater than its own principal diagonal term when the product-determinant is obtained by multiplying row by row*, i.e.

$$\ngtr (a_1^2+a_1'^2+a_2^2+a_2'^2+a_3^2+a_3'^2)(b_1^2+b_1'^2+b_2^2+b_2'^2+b_3^2+b_3'^2)(c_1^2+c_1'^2+c_2^2+c_2'^2 + c_3^2+c_3'^2).$$

7. From § 4 it is seen that the limit $s_1^2s_2^2s_3^2$ which $|\mu|^2$ cannot exceed will actually be reached when for $r=1, 2, 3$

$$\begin{vmatrix} a_r+a'_ri & b_r+b'_ri \\ A_r-A'_ri & B_r-B'_ri \end{vmatrix} \cdot \begin{vmatrix} a_r-a'_ri & b_r-b'_ri \\ A_r+A'_ri & B_r+B'_ri \end{vmatrix}$$
$$+\begin{vmatrix} a_r+a'_ri & c_r+c'_ri \\ A_r-A'_ri & C_r-C'_ri \end{vmatrix} \cdot \begin{vmatrix} a_r-a'_ri & c_r-c'_ri \\ A_r+A'_ri & C_r+C'_ri \end{vmatrix}$$
$$+\begin{vmatrix} b_r+b'_ri & c_r+c'_ri \\ B_r-B'_ri & C_r-C'_ri \end{vmatrix} \cdot \begin{vmatrix} b_r-b'_ri & c_r-c'_ri \\ B_r+B'_ri & C_r+C'_ri \end{vmatrix} = 0.$$

This can only happen when one of the factors of each term of the left-hand member vanishes: and as the vanishing of one factor implies the vanishing of the co-factor, it can only happen when

$$(\text{X.})\quad \begin{cases} \dfrac{a_1+a'_1 i}{A_1-A'_1 i}=\dfrac{b_1+b'_1 i}{B_1-B'_1 i}=\dfrac{c_1+c'_1 i}{C_1-C'_1 i}=\varsigma_1 \text{ say,} \\ \dfrac{a_2+a'_2 i}{A_2-A'_2 i}=\dfrac{b_2+b'_2 i}{B_2-B'_2 i}=\dfrac{c_2+c'_2 i}{C_2-C'_2 i}=\varsigma_2 \text{ say,} \\ \dfrac{a_3+a'_3 i}{A_3-A'_3 i}=\dfrac{b_3+b'_3 i}{B_3-B'_3 i}=\dfrac{c_3+c'_3 i}{C_3-C'_3 i}=\varsigma_3 \text{ say;} \end{cases}$$

in other words, when the elements of each column of μ are proportional to the elements of the corresponding column of M′. But when this is the case, each element of μ can be replaced by a multiple of the corresponding element of M′, with the result that we shall have

$$\mu=\varsigma_1\varsigma_2\varsigma_3.\mathrm{M}'=\varsigma_1\varsigma_2\varsigma_3.(\mu')^{3-1};$$

also, the column-by-column multiplication of μ by μ' would give

$$\begin{vmatrix} \varsigma_1\mu' & \cdot & \cdot \\ \cdot & \varsigma_2\mu' & \cdot \\ \cdot & \cdot & \varsigma_3\mu' \end{vmatrix},$$

from which the same deduction could be made.

Similarly the limit

$$(a_1^2+a_1'^2+a_2^2+a_2'^2+a_3^2+a_3'^2)(b_1^2+b_1'^2+b_2^2+b_2'^2+b_3^2+b_3'^2)(c_1^2+c_1'^2+c_2^2+c_2'^2 + c_3^2+c_3'^2)$$

will be reached when the elements of each *row* of μ are proportional to the elements of the corresponding row of M′, and row-by-row multiplication will give

$$\mu\mu'=\begin{vmatrix} \varsigma'_1\mu' & \cdot & \cdot \\ \cdot & \varsigma'_2\mu' & \cdot \\ \cdot & \cdot & \varsigma'_3\mu' \end{vmatrix}.$$

Both limits will be reached, and will therefore coalesce when *all* the elements of μ are proportional to the corresponding elements of M′, and row-by-row multiplication will then give the same result as column-by-column multiplication, namely

$$\mu\mu'=\begin{vmatrix} \varsigma\mu' & \cdot & \cdot \\ \cdot & \varsigma\mu' & \cdot \\ \cdot & \cdot & \varsigma\mu' \end{vmatrix}.$$

In this event we may appropriately speak of the determinant *having a maximum value* or *being a maximum determinant.* Such is evidently possible when μ is axisymmetric or axially skew, because then the two limits are identical. The same also is true when the elements of μ are equimodular.

8. When the elements of μ are proportional to the corresponding elements of M′, the elements of μ' are proportional to the corresponding elements of M, and therefore the moduli of the elements of μ are proportional to the moduli of the corresponding elements of M. It thus follows that when the elements of μ are proportional to the corresponding elements of M′ and the elements of μ (or M) are equimodular, then the elements of M (or μ) are equimodular also. By rationalising the denominator of each ratio in (X.), it is thus seen that when the elements of μ are proportional to the corresponding elements of M′ and are equimodular, the product of any element of μ by the corresponding element of M is constant, or, in Sylvester's language, μ is "inversely orthogonal." Also, if μ be "inversely orthogonal" and have equimodular elements, the elements must be proportional to the elements of M′, and therefore by a preceding result $|\mu|$ must have its maximum value. The problem of finding inversely-orthogonal determinants is thus closely connected with the problem of finding determinants of maximum value. Any results, therefore, obtained by Sylvester in his efforts towards a solution of the former problem deserve attention in the present connection, our scrutiny being all the closer because of the fact that his assertions are not always accompanied by proof.

9. Taking the very special form of determinant which represents the difference-product of $z, y, x, w, \ldots$, namely, the alternant $|z^0y^1x^2w^3\ldots|$, let us inquire if there be values of $z, y, x, w, \ldots$, which make it inversely orthogonal.

On multiplying each element of $|z^0y^1x^2w^3|$ by the corresponding element of the adjugate determinant we obtain the array

$$\begin{array}{rrrr}
yxw\,|y^0x^1w^2|, & -z\,|y^0x^2w^3|, & z^2\,|y^0x^1w^3|, & -z^3\,|y^0x^1w^2| \\
-zxw\,|z^0x^1w^2|, & y\,|z^0x^2w^3|, & -y^2\,|z^0x^1w^3|, & y^3\,|z^0x^1w^2| \\
zyw\,|z^0y^1w^2|, & -x\,|z^0y^2w^3|, & x^2\,|z^0y^1w^3|, & -x^3\,|z^0y^1w^2| \\
-zyx\,|z^0y^1x^2|, & w\,|z^0y^2x^3|, & -w^2\,|z^0y^1x^3|, & w^3\,|z^0y^1x^2|,
\end{array}$$

and the condition for inverse-orthogonalism is that all the elements of this array be equal. Now the equality of the first and second elements of any row of the array is tantamount to the vanishing of Σzyx, the equality of the second and third elements to the vanishing of Σzy, and the equality of the third and fourth to the vanishing of Σz: consequently z, y, x, w must be the roots of an equation of the form $\omega^4 = a$. Again, the equality of the elements of the first *column* is tantamount to the vanishing of

$$(z^4 - y^4)/(z - y), \quad (y^4 - x^4)/(y = x), \quad (x^4 - w^4)/(x - w):$$

so that, since z, y, x, w must from the nature of the problem be all different, the equality of the elements of the first column is tantamount to the equality of z^4, y^4, x^4, w^4—a result not different from that obtained in dealing with the rows. Our conclusion thus is that *the alternant* $|\, z^0 y^1 x^2 w^3 \ldots |$ *of the* n*th order will be inversely orthogonal if* z, y, x, w, ... *be the roots of the equation* $\omega^n = a$.

10. Since a determinant that is inversely orthogonal (an *ant-orthogonant* say) continues to be so when the elements of any row or column are all multiplied by the same quantity, we may without loss of generality make $a=1$, $z, y, x, \ldots$ then becoming the nth roots of unity. Further, by taking ς to be a *primitive* nth root of unity $z, y, x, \ldots$ then become $\varsigma, \varsigma^2, \varsigma^3, \ldots$, and we see that the ant-orthogonant thus reached may be written

$$\begin{vmatrix} 1 & \varsigma & \varsigma^2 & \varsigma^3 & \ldots & \varsigma^{n-1} \\ 1 & \varsigma^2 & \varsigma^4 & \varsigma^6 & \ldots & \varsigma^{2n-2} \\ 1 & \varsigma^3 & \varsigma^6 & \varsigma^9 & \ldots & \varsigma^{3n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \varsigma^{n-1} & \varsigma^{2n-2} & \varsigma^{3n-3} & \ldots & \varsigma^{(n-1)^2} \\ 1 & 1 & 1 & 1 & \ldots & 1 \end{vmatrix}.$$

By passing the last row over all the others the result becomes axisymmetric, and is then identical with that obtained by using Sylvester's "rule."

Since, in addition, the elements are all unimodular, the determinant reached is also, by a result of § 6, an instance of a maximum determinant.

11. If μ be Sylvester's ant-orthogonant of the nth order, it is readily found by determinant multiplication that

$$\mu^2 = n^n.(-1)^{\frac{1}{2}(n-1)(n-2)}.$$

And as we already know that

$$\mu\mu' = n^n,$$

it follows that

$$\mu = \mu'.(-1)^{\frac{1}{2}(n-1)(n-2)}.$$

It may also be worth noting that the complementary minor of the first element is symmetric with respect to both diagonals.

12. If μ_r, μ_s be maximum determinants of the rth and sth orders respectively, and μ_{rs} the maximum determinant of the (rs)th order formed according to Sylvester's second rule, then

$$\mu_{rs} = (\mu_r)^s(\mu_s)^r.$$

13. It would, of course, be unwise to conclude without further investigation that the determinant reached in § 9 is the only n-line orthogonant. As an illustration, let us inquire whether the axisymmetric determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & a & b & c \\ 1 & b & d & e \\ 1 & c & e & f \end{vmatrix}$$

can only be inversely orthogonal when

$$a,\ b,\ c,\ d,\ e,\ f = -i,\ (-i)^2,\ (-i)^3,\ (-i)^4,\ (-i)^6,\ (-i)^9,$$
$$= -i,\ -1,\ i,\ 1,\ -1,\ -i.$$

To ensure inverse-orthogonalism the products which must be equal are

$$\begin{array}{ll} adf+2bce-c^2d-b^2f-e^2a, & b(-bf-c-e+b+f+ce), \\ -df-ce-be+cd+bf+e^2, & c(be+c+d-b-e-cd), \\ bf+c^2+ae-bc-af-ce, & d(af+2c-a-f-c^2), \\ -be-bc-ad+b^2+ae+cd, & e(-ae-c-b+a+e+bc), \\ a(df+2e-d-f-e^2), & f(ad+2b-a-d-b^2)\,; \end{array}$$

and since the equality of the 8th and 10th products is tantamount to the equality

$$\frac{d}{f}=\frac{2b-b^2-a}{2c-c^2-a},$$

it is evident that the said two products will be equal if we make $c=b$ and $f=d$. Doing this we next see that the equality of the 3rd and 6th products is tantamount to

$$a(e-d)=b^2(e-d)\,;$$

and, since the taking of $e=d$ is excluded by the fact that this would cause both products to vanish, we are forced, in order that the two may be equal, to take $a=b^2$. It will be found, however, that this taking of $a,\ c,\ f=b^2,\ b,\ d$ makes certain others of the products equal—that, in fact, there only remain five to be dealt with, these now taking the forms

$$b^2(d-e)(d+e-2),\quad -(d-e)(d+e-2b),\quad (d-e)b(1-b),$$
$$d(b-1)(bd+d-2b),\quad -e(b-1)(be+e-2b).$$

Recalling again the fact that none of the factors here visible can be allowed to vanish, we see that the equality of the first three products is tantamount to

$$d+e=2b(b+1)/(b^2+1)=b^2+b,$$

and that therefore the said products can only be properly equal when

$$b=-1,\quad e=-d.$$

As, however, their common value is then $-4d$, and as the 4th and 5th products have this value also, our final result is that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & d & -d \\ 1 & -1 & -d & d \end{vmatrix}$$

is inversely orthogonal whatever d *may be, the product of any element by its* *co-factor in the determinant being* $-4d$, *and the value of the determinant itself being consequently* $-16d$.*

When d is a complex quantity whose modulus is 1, the determinant is both inversely orthogonal and equimodular, and therefore is a determinant of maximum value. This was first pointed out by Hadamard.

14. If in addition to requiring the elements of μ to be unimodular we insist on them being real—in other words, if we seek to construct maximum determinants whose elements are $+1$ or -1—we soon find that the problem is soluble only for certain orders of determinants. We can show, however, that *if a solution be obtained for order* r *it is easy to give a solution for order* 2r. For the rows of the r-line determinant being A, B, C, ... we know that

$$\begin{array}{llll} AB=0, & AC=0, & AD=0, & \ldots\ldots \\ & BC=0, & BD=0, & \ldots\ldots \\ & & CD=0, & \ldots\ldots \\ & & & \ldots\ldots \end{array}$$

and this being the case the $2r$-line determinant whose rows are

$$(A, A), \quad (A, -A), \quad (B, B), \quad (B, -B), \quad (C, C), \quad (C, -C), \ldots\ldots$$

has evidently the same property. Thus, the determinant for the 2nd order being

$$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix},$$

the determinant for the 4th order is

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix},$$

which agrees with the result of § 13.

The determinants of order 2^m thus obtainable are all axisymmetric.

15. Hadamard's 12-line determinant of this kind has also a latent axisymmetry which it is preferable to put in evidence. If we denote a row of three elements by the place-numbers of those which are negative, thus

$$1 \quad 1 \; -1 \text{ by } 3, \quad -1 \quad 1 \; -1 \text{ by } 13, \quad \ldots\ldots$$

* Besides the solution obtained in this paragraph there are at least two others, the set of three being

$$\begin{aligned} a, b, c, d, e, f &= 1, \ -1, \ -1, \ x, \ -x, \ x, \\ &= x, \ -x, \ -1, \ x, \ -1, \ 1, \\ &= x, \ -1, \ -x, \ 1, \ -1, \ x. \end{aligned}$$

They are not, however, really different.

the axisymmetric determinant in question is

.	.	.	.
.	.	123	123
.	123	.	123

....

3	12	23	1
3	13	12	3
3	23	13	2

....

2	23	12	1
2	12	13	3
2	13	23	2

....

23	1	1	12
23	3	3	13
23	2	2	23.

This notation is very useful in that, if we wish to test whether the product of two different rows vanishes (as every such product ought), we have only to count the number of different digits in each of the four sections of the two rows; thus, in the case of the rows

.	123	.	123
23	2	2	23

there are 2 corresponding digits different in the first section, 2 in the second, 1 in the third, and 1 in the fourth—that is to say, 6 altogether, which give -1 on performing multiplication, thus making the product $6-6$. Similarly in the case of the last two rows the like number is $0+2+2+2$.

16. On the other hand, Hadamard's 20-line determinant appears to be essentially unsymmetric. As the result of a fresh investigation, in which axisymmetry was steadily kept in view, the following has been reached—

.	.	.	.
.	.	12345	12345
.	12345	.	12345
45	145	135	24
45	245	234	15

....

34	123	345	45
35	123	123	12
3	1245	1245	3
345	34	24	234
345	35	15	135

....

24	234	125	14
25	235	134	34
245	12	45	123
25	134	235	35
24	135	124	25
......	......	......	......
235	25	25	245
234	24	13	235
23	345	345	12
234	15	23	134
235	14	14	145.

Here the axisymmetry is first departed from in the case of the elements in the places (9, 11), (11, 9).

17. When the elements of μ are real we see from (X.) of § 7 that μ will simultaneously reach its row-by-row limit and its column-by-column limit when

$$\begin{aligned}\frac{a_1}{A_1}=\frac{b_1}{B_1}=\frac{c_1}{C_1}\\ =\frac{a_2}{A_2}=\frac{b_2}{B_2}=\frac{c_2}{C_2}\\ =\frac{a_3}{A_3}=\frac{b_3}{B_3}=\frac{c_3}{C_3};\end{aligned}$$

in other words, when the elements of μ are proportional to the corresponding elements of the adjugate determinant. This property, we know, is possessed by an orthogonant, the common ratio, in the case where the name orthogonant is strictly applicable, being the orthogonant itself. A set of interesting examples is to be found in the series of skew determinants*

$$\begin{vmatrix} a & b \\ -b & a \end{vmatrix}, \quad \begin{vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{vmatrix}, \quad \begin{vmatrix} a & b & c & d & e & f & g & h \\ -b & a & d & -c & f & -e & -h & g \\ -c & -d & a & b & g & h & -e & -f \\ -d & c & -b & a & h & -g & f & e \\ -e & -f & -g & -h & a & b & c & d \\ -f & e & -h & g & -b & a & -d & c \\ -g & h & e & -f & -c & d & a & -b \\ -h & -g & f & -e & -d & -c & b & a \end{vmatrix}, \ldots$$

* Another form of the second of the series, which looks essentially different, is not really so, the one O_2' being obtainable from the other O_2 by altering the signs of the 1st row and 1st column, and then changing columns into rows. As regards the third of the series, its first quarter is O_2', the quarter to the right is of the same type as O_2 but with the signs of three rows changed, the last quarter is O_2', and the remaining quarter is of the same type as O_2' with the signs of one row changed: other forms, however, are obtainable, the first quarter being always O_2 or O_2', and all the other elements being determinable when the signs of the places (2, 5), (2, 7), (3, 5) are known.

or, say, O_2, O_4, O_8, ..., the second of which has for its adjugate

$$\begin{vmatrix} a\Sigma a^2 & b\Sigma a^2 & c\Sigma a^2 & d\Sigma a^2 \\ -b\Sigma a^2 & a\Sigma a^2 & d\Sigma a^2 & -c\Sigma a^2 \\ -c\Sigma a^2 & -d\Sigma a^2 & a\Sigma a^2 & b\Sigma a^2 \\ -d\Sigma a^2 & c\Sigma a^2 & -b\Sigma a^2 & a\Sigma a^2 \end{vmatrix},$$

and for its attained upper limit $(\Sigma a^2)^2$.

18. Bringing together the results of §§ 7, 17, we note that a maximum determinant with *complex* elements is an *ant-orthogonant* only when the elements are equimodular, and that a maximum determinant with *real* elements is always an *orthogonant*. In the latter case, however, it has to be noticed that if the only elements permissible be $+1$ and -1 the distinction between orthogonant and ant-orthogonant disappears, because then

$$a_{rs}A_{rs} = a_{rs}^2\frac{A_{rs}}{a_{rs}} = \frac{A_{rs}}{a_{rs}}.$$

CAPE TOWN,
Dec. 28, 1908.

[19. Added 21/1/09.] As the last step of the reasoning in §5 may not carry conviction to some, I append, on Mr. Hough's suggestion, another proof of the most direct and simple character:—

THEOREM.—The row-by-row product of two determinants whose corresponding elements are complex conjugates is not greater than its own principal diagonal term.

PROOF.—Let the two determinants and their product be

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots \\ b_1 & b_2 & b_3 & \dots \\ c_1 & c_2 & c_3 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}, \quad \begin{vmatrix} a_1' & a_2' & a_3 & \dots \\ b_1' & b_2 & b_3' & \dots \\ c_1' & c_2' & c_3' & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}, \quad \begin{vmatrix} (aa') & (ab') & (ac') & \dots \\ (ba') & (bb') & (bc') & \dots \\ (ca') & (cb') & (cc') & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}.$$

Then in the first place it is clear that

$$\begin{vmatrix} (aa') & (ab') \\ (ba') & (bb') \end{vmatrix} \not> (aa')(bb') \qquad \text{(I.)}$$

because (ab'), (ba') are complex conjugates. In the second place, from a well-known property of determinants we have

$$\begin{vmatrix} (aa') & (ab') & (ac') \\ (ba') & (bb') & (bc') \\ (ca') & (cb') & (cc') \end{vmatrix} = \begin{Vmatrix} \begin{vmatrix} (aa') & (ab') \\ (ba') & (bb') \end{vmatrix} & \begin{vmatrix} (aa') & (ac') \\ (ba') & (bc') \end{vmatrix} \\ \begin{vmatrix} (aa') & (ab') \\ (ca') & (cb') \end{vmatrix} & \begin{vmatrix} (aa') & (ac') \\ (ca') & (cc') \end{vmatrix} \end{Vmatrix} \div (aa'),$$

∴ by (I.)

$$\not> \begin{vmatrix} (aa') & (ab') \\ (ba') & (bb') \end{vmatrix} \cdot \begin{vmatrix} (aa') & (ac') \\ (ca') & (cc') \end{vmatrix} \div (aa'),$$

and again by (I.)

$$\not> (aa')(bb')(cc'). \qquad \text{(II.)}$$

Similarly P, Q, R, S being the co-factors of the elements (dd'), (dc'), (cd'), (cc') in the four-line determinant we have

$$\begin{vmatrix} (aa') & (ab') & (ac') & (ad') \\ (ba') & (bb') & (bc') & (bd') \\ (ca') & (cb') & (cc') & (cd') \\ (da') & (db') & (dc') & (dd') \end{vmatrix} = \begin{vmatrix} \mathrm{P} & \mathrm{Q} \\ \mathrm{R} & \mathrm{S} \end{vmatrix} \div \begin{vmatrix} (aa') & (ab') \\ (ba') & (bb') \end{vmatrix},$$

∴ by (I.)

$$\ngtr \ \mathrm{PS} \div \begin{vmatrix} (aa') & (ab') \\ (ba') & (bb') \end{vmatrix},$$

∴ by (II.)

$$\ngtr \frac{\begin{vmatrix} (aa') & (ab') \\ (ba') & (bb') \end{vmatrix} \cdot \begin{vmatrix} (aa') & (ac') \\ (ca') & (cc') \end{vmatrix}}{(aa')} \cdot (aa')(bb')(dd') \div \begin{vmatrix} (aa') & (ab') \\ (ba') & (bb') \end{vmatrix},$$

and again by (I.)

$$\ngtr (aa')(bb')(cc')(dd').$$

And so generally.